Journal of Applied Mechanics and Technical Physics, Vol. 38, No. 5, 1997

STATIONARY PLASMA FLOW IN A MAGNETIC FIELD IN A NOZZLE WITH A VACUUM GAP NEAR THE WALLS

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In the problem of controlled thermonuclear fusion, great attention has been paid to investigations of strong nonlinear waves and plasma flows [1]. For plasma heating, Buiko et al. [2, 3] proposed using a shock wave that emerges upon quasi-stationary outflow of a magnetized plasma through a nozzle. In this connection, studying stationary plasma flows in nozzles in the presence of a magnetic field is very important [1].

The present paper deals with a stationary axisymmetric flow of a magnetized plasma in a nozzle. In contrast to the previous papers, it is assumed here that there is a vacuum gap between the nozzle walls and the plasma, in which there is a magnetic field. To clarify how the presence of the gap affects the plasma motion, we consider, without allowance for other factors, the simplest single-fluid magnetohydrodynamic (MHD) model of an ideally conducting incompressible plasma. The nozzle walls are assumed to be ideally conducting as well. In addition, an approximation that is long-wave along the axis of symmetry is used. It turns out that even in such a model, qualitatively new effects caused by the presence of a gap with a magnetic field in it appear.

1. Formulation of the Problem. An ideal (inviscid, non-heat-conducting, and possessing infinite conductivity) incompressible plasma is considered. The flow is considered stationary and axisymmetric. The cylindrical coordinate system (r, φ, z) with the z axis directed along the axis of symmetry is introduced. The plasma occupies the region $z \geqslant 0$, $r \leqslant r_1(z)$, where r_1 is the radius of the free plasma surface. The flow parameters are specified for $z = 0$. Its evolution is studied, depending on the z coordinate.

To make a transition to dimensionless quantities, we introduce the scales of length, velocity, and density. The characteristic scale of changes along the z axis is taken as a unit length, the characteristic magnitude of the axial velocity component for $z = 0$ serves as a unit velocity, and the plasma density is assumed to be equal to unity. The magnetic field is nondimensionalized in density and characteristic velocity. In what follows, all quantities are used in nondimensional form, unless otherwise specified.

We use the following notation: (u, v, w) and (H_1, H_2, H_3) are the velocity and magnetic-field components corresponding to (r, φ, z), A is the azimuthal component of the vector potential, p is the pressure, δ is the nondimensional value of r_0 for $z = 0$ [r_0 is the nozzle radius (a known function of z)]. It is assumed that $\delta \ll 1$. Owing to the axial symmetry inside the pinch, it is assumed that $v = 0$ and $H_2 = 0$. In the general case, $H_2 \neq 0$ in the gap.

In going to a long-wave approximation, the coordinates and the functions are extended over the relations $r^2 \to \delta^2\eta$, $z \to z$, $2ur \to \delta^2 q$, $w \to w$, $p \to p$, $2H_1 r \to \sqrt{4\pi}\delta^2 h$, $H_3 \to \sqrt{4\pi}H$, and $2rA \to \sqrt{4\pi}\delta^2 a$. The boundaries $r_1(z)$ and $r_0(z)$ transform into $\eta_1(z)$ and $\eta_0(z)$. Note that $\eta_0(0) = 1$ according to the definition of δ.

Within the framework of single-fluid ideal magnetic hydrodynamics, with allowance for the axial symmetry, the plasma motion is described by the following equations:

$$\delta^2(qq_\eta - q^2/(2\eta) + wq_z) = -4\eta P_\eta + \delta^2(hh_\eta - h^2/(2\eta) + Hh_z),$$

$$qw_\eta + ww_z = -P_z + hH_\eta + HH_z, \qquad w_z + q_\eta = 0, \qquad wa_z + qa_\eta = 0, \tag{1.1}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 38, No. 5, pp. 23–28, September–October, 1997. Original article submitted February 6, 1996; revision submitted March 25, 1996.

0021-8944/97/3805-0673 $18.00

$$h = -a_z, \qquad H = a_\eta, \qquad P = p + H^2/2 + \delta^2 h^2/(8\eta)$$

(the subscripts denote partial differentiation with respect to the corresponding independent variables).

As the boundary conditions, we adopt the following ones: for $\eta = 0$,

$$q = h = 0; \tag{1.2}$$

on the free surface ($\eta = \eta_1$), the normal component of the magnetic field equals zero, the quantity P is continuous, the kinematic condition

$$a_z + a_\eta \eta_{1z} = 0; \tag{1.3}$$

$$q = w\eta_{1z} \tag{1.4}$$

is satisfied, and the normal component of the magnetic field is assumed to be zero at the tube walls as well.

The equations for a magnetic field in a vacuum gap are of the form

$$4(\eta H_{*\eta})_\eta + \delta^2 H_{*zz} = 0, \qquad 4\eta h_{*\eta\eta} + \delta^2 h_{*zz} = 0, \qquad æ = \text{const}, \tag{1.5}$$

where H_* and h_* are the values of H and h in the gap, $\sqrt{4\pi}\delta æ = H_{*2} r$, and H_{*2} is the azimuthal component of the magnetic field in the gap.

In going to a long-wave approximation, the terms in (1.1) and (1.5) that are proportional to δ^2 are omitted. As a result, from (1.5) we obtain $H_* = H_*(z)$. Using the assumptions of the magnetic field at the tube walls and of the free surface, we find that $H_* = \Phi/(\eta_0 - \eta_1)$, where $\Phi = \text{const}$. With allowance for the smallness of δ^2, for $\eta = \eta_1$, the continuity condition of P takes the form

$$P = \Phi^2/[2(\eta_0 - \eta_1)^2] + æ^2/(2\eta_1). \tag{1.6}$$

After the terms with δ^2 are ignored, the equations derived from (1.1) are transformed similarly [4]. New independent variables z' and ν ($\nu \in [0,1]$) are introduced into the relations $z = z'$ and $\eta = R(z',\nu)$, where R satisfies the equations and the boundary conditions

$$wR_{z'} = q; \quad R(z',0) = 0, \quad R(z',1) = \eta_1, \quad R(0,\nu) = \eta_{10}\nu,$$

where η_{10} is the value of η_1 for $z' = 0$. In this case, the unknown boundary $\eta_1(z)$ transforms into the known $\nu = 1$.

For differential operators, we write

$$\frac{\partial R}{\partial \nu}\frac{\partial}{\partial z} = \frac{\partial R}{\partial \nu}\frac{\partial}{\partial z'} - \frac{\partial R}{\partial z'}\frac{\partial}{\partial \nu}, \qquad \frac{\partial R}{\partial \nu}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \nu}.$$

After the terms with δ^2 are ignored and q is discarded, in variables z' and ν, system (1.1) takes the following form (below, the prime at z' is omitted):

$$P_\nu = 0, \quad ww_z = -P_z + HH_z, \quad (wR_\nu)_z = 0, \quad wa_z = 0, \quad hR_\nu = R_z a_\nu, \quad HR_\nu = a_\nu. \tag{1.7}$$

The above relation incorporates that $P_\nu = 0$ and $a_z = 0$. We assume that $w \neq 0$. From the equation $wa_z = 0$, it follows that $a = a(\nu)$. It is assumed that $a = a(\nu)$. With allowance for $a = a(\nu)$, $\eta_1(z) = R(z,1)$, and the definition of R, the boundary conditions (1.2)–(1.4) are satisfied automatically. Using (1.6), we exclude P from (1.7). The equations are integrated over z from 0 to z and are solved with respect to η_0. As a result, we obtain

$$\eta_0 = R_1 + \Phi\Big[\frac{\Phi^2}{(1-R_{10})^2} - \psi - \frac{æ^2}{R_1} + \frac{æ^2}{R_{10}} + \Big(\frac{a_\nu}{R_1}\Big)^2 - \Big(\frac{a_\nu}{R_{10}}\Big)^2\Big]^{-1/2},$$
$$R(\nu,z) = \int_0^\nu w_0 R_{10}(w_0^2 + \psi)^{-1/2} d\nu, \quad R_1 = R(1,z), \quad R_{10} = R(1,0), \tag{1.8}$$

where $\psi = w^2 - w_0^2$ and $w_0(\nu)$ is the value of w for $z = 0$. Equations (1.8) express an implicit dependence $\psi(z,\nu)$ that should be found.

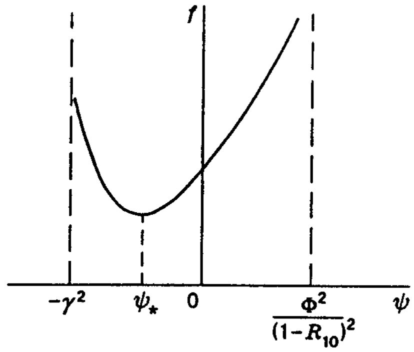

Fig. 1

To reveal unambiguously the effect of a vacuum gap with a magnetic field on plasma flow, we shall first analyze (1.8) for the case where $æ = 0$ and $a_\nu = 0$. This implies that the azimuthal field component is zero and there is no magnetic field inside the pinch. After that, we shall analyze the case where $æ \neq 0$ and $a_\nu \neq 0$.

Proposition. *Assume that $æ = 0$ and $a_\nu = 0$, the function $w_0(\nu)$ is bounded $[w_0(\nu) \geqslant \gamma > 0]$, and γ is a constant,*

$$2\lambda = \frac{(1-R_{10})^3}{\Phi^2} - R_{10}\int_0^1 w_0^{-2}\,d\nu,$$

$$2\lambda_1 = \Phi\left[\frac{\Phi^2}{(1-R_{10})^2} + \gamma^2\right]^{-3/2} - \int_0^1 w_0 R_{10}(w_0^2 - \gamma^2)^{-3/2}\,d\nu.$$

It is assumed that $\lambda_1 < 0$. The quantity ψ is considered in the interval

$$-\gamma^2 < \psi < \Phi^2/(1-R_{10})^2.$$

The following cases are distinguished:

1. *$\lambda > 0$. The solution then exists for any $\eta_0 > \eta_{0*}$ and $\eta_{0*} < 1$, where η_{0*} is the solution of a certain integral equation, ψ and w increase as η_0 increases from η_{0*} to ∞, whereas R_1 decreases to finite values for $\eta_0 \to \infty$.*

2. *$\lambda < 0$. The solution exists for $\eta_{0*} < \eta_0 < \eta_0(l)$, $\eta_{0*} < 1$, and $\eta_0(l) > 1$:*

$$\eta_0(l) = \Phi\left[\frac{\Phi^2}{(1-R_{10})^2} + \gamma^2\right]^{-1/2} + \int_0^1 w_0 R_{10}(w_0^2 - \gamma^2)^{-1/2} d\nu, \tag{1.9}$$

ψ and w decrease as η_0 increases from η_{0*} to $\eta_0(l)$, whereas R_1 increases.

In both cases, for $\eta \to \eta_{0*}$ the derivatives ψ and R_1 with respect to η_0 in magnitude tend to infinity.

Proof. We set $æ = 0$ and $a_\nu = 0$ in (1.8). Then, it follows from (1.8) that ψ is not dependent on ν. We consider the right-hand side of (1.8) as a function of ψ and denote it as $f(\psi)$ (R_1 is regarded as a function of ψ as well). It then follows from (1.8) that $\psi = \psi(\eta_0)$. We shall analyze this dependence.

In the interval indicated above, $f(\psi) > 0$. It is easy to see that $f''(\psi) > 0$. It follows that $f'(\psi)$ is a monotonically decreasing function. From the expression for $f'(\psi)$, we have $f'(-\gamma^2) = \lambda_1 < 0$ and $f'(0) = \lambda$.

(1) Let $\lambda > 0$. By virtue of a monotone increase in $f'(\psi)$ and of the inequality $f'(-\gamma^2) < 0$, the function $f(\psi)$ is of the form shown qualitatively in Fig. 1. Clearly, if η_0 increases (the nozzle diverges), then $f(\psi)$, ψ, and, hence, w increase and the solution exists for any $\eta_0 > 1$; if η_0 decreases (the nozzle converges), then $f(\psi)$, ψ, and, hence, w decrease and the solution exists only for $\eta_0 \geqslant \eta_{0*}$ and $\psi \geqslant \psi_*$, where ψ_* is found from the equation $f'(\psi_*) = 0$, and $\eta_{0*} = f(\psi_*)$.

(2) Let $\lambda < 0$. In this case, $f'(0) < 0$, and the function $f(\psi)$ has a minimum to the right of the coordinate origin. It follows that if η_0 increases, then $f(\psi)$ also increases, and ψ and w decrease, the solution

existing until ψ reaches the value $-\gamma^2$. Thus, the solution exists for $\eta_0 \leqslant \eta_0(l)$, where $\eta_0(l)$ is expressed according to (1.9). It follows from the definition of w that, at $\eta_0 = \eta_0(l)$, we have $w = 0$ for ν specified by the equation $w_0(\nu) = \gamma$. If η_0 decreases, then $f(\psi)$ also decreases, while ψ and w increase, the solution existing for $\eta_0 \geqslant \eta_{0*}$ and $\psi \leqslant \psi_*$, where ψ_* is the solution of the equation $f'(\psi_*) = 0$.

Evidently, in both cases the derivatives of ψ with respect to η_0 in magnitude tend to ∞ for $\eta_0 \to \eta_{0*}$ and $\psi \to \psi_*$. The dependence of R_1 on η_0 follows from a monotone decrease of R_1 as a function of ψ.

For mathematical generality, we analyze the case $\lambda_1 > 0$. Physically, such a flow is far from reality, because under this condition the function $(w_0^2 - \gamma^2)^{-3/2}$ should be integrated on $[0,1]$. Let the conditions of the proposition be satisfied, except for the inequality $\lambda_1 < 0$. We assume that $\lambda_1 \geqslant 0$. Then the solution exists for any $\eta_0 \geqslant \eta_0(l)$, ψ growing as η_0 increases.

The proof follows from the fact that the function $f(\psi)$ for $\lambda_1 \geqslant 0$ on the section $[-\gamma^2, \Phi^2(1-R_{10})^{-2}]$ is monotone.

Thus, it follows from the Proposition that the dependence of ψ, w, and R_1 on η_0 is qualitatively different, depending on the sign of λ. By analogy with flows of shallow water or gas, we call the flows with $\lambda < 0$ subcritical and those with $\lambda > 0$ supercritical. In a subcritical regime, the axial velocity decreases in a divergent nozzle and increases in a convergent one, and vice versa in a supercritical regime.

2. Allowance for the Azimuthal Component of the Magnetic Field in a Gap and of the Field Inside a Plasma. Let us analyze how the indicated parameters exert an effect on the flow pattern. We first set $a_\nu = 0$ and $æ \neq 0$ in (1.8). In this case, ψ is not dependent, as before, on ν, and one can consider R_1 as a function of ψ. We introduce the following notation:

$$g(\psi) = \Phi^2(1 - R_{10})^{-2} - \psi - æ^2/R_1 + æ^2/R_{10}, \qquad g_1(\psi) = \Phi^2(1 - R_{10})^{-2} - \alpha_1\psi,$$
$$G_1(\psi) = \Phi^2(1 - R_{10})^{-2} - \alpha_2\psi, \qquad G_2(\psi) = \Phi^2(1 - R_{10})^{-2} - \alpha_3\psi,$$

$g_2(\psi) = G_1$ if $\psi \geqslant 0$ and $g_2(\psi) = G_2$ if $\psi < 0$, $f = R_1 + \Phi/g^{1/2}$, $f_1 = R_1 + \Phi/g_1^{1/2}$, $f_2 = R_1 + \Phi/g_2^{1/2}$, $F_1 = R_1 + \Phi/G_1^{1/2}$, $F_2 = R_1 + \Phi/G_2^{1/2}$, and α_1 is determined by the condition $g_1'(0) = g'(0)$ and α_2 is determined by the condition $G_1(\psi_0) = g(\psi_0) = 0$ [such a ψ_0 exists by virtue of the continuity and monotonicity of $g(\psi)$], and α_3 is determined by the condition $G_2(-\gamma^2) = g(-\gamma^2)$.

Let us analyze the properties of the function $g(\psi)$ on the section $-\gamma^2 \leqslant \psi \leqslant \psi_0$. Having differentiated it twice, we obtain

$$g'(\psi) < -1,$$
$$4g''(\psi) = -2æ^2 R_1^{-3}\Big(\int_0^1 w_0 R_{10}(w_0^2 + \psi)^{-3/2}\,d\nu\Big)^2 + 3æ^2 R_1^{-2}\int_0^1 w_0 R_{10}(w_0^2 + \psi)^{-5/2}\,d\nu.$$

Using the Cauchy–Bunyakovskii inequality, there is no difficulty in obtaining that $g''(\psi) > 0$. Thus, $g(\psi)$ is a concave monotonically decreasing positive function. From that and also from the definition of g_1 and g_2 follow the relations $g_1 \leqslant g \leqslant g_2$, $\alpha_1 = g'(0)$, and $\alpha_3 < g'(0) < \alpha_2 < -1$. Then

$$f_2(\psi) \leqslant f(\psi) \leqslant f_1(\psi), \tag{2.1}$$
$$f'(0) = f_1'(0), \quad f'(0) > f_2'(+0), \quad f'(0) < f_2'(-0),$$

where $f'(\pm 0)$ is the right or left derivative f_2 in zero.

It is seen that the results of the proposition hold true for the functions f_1, F_1, and F_2 with replacement of λ by the zero derivative of the corresponding function. It follows that the qualitative form of f_1, F_1, and F_2 is similar to that of $f(\psi)$ (see Fig. 1). Owing to the fact that f is constrained by inequalities (2.1), the specific features of the flow will be qualitatively the same as in the case considered, i.e., for $æ \neq 0$. Correspondingly, the subcritical and supercritical flow conditions take the form $f_2'(-0) < 0$ and $f_2'(+0) > 0$. With satisfaction of these conditions, the qualitative similarity of the solutions with $æ = 0$ and $æ \neq 0$ is guaranteed. For physical estimates, it suffices to preserve either the criterion $f'(0) > 0$ or the criterion $f'(0) < 0$, since, in this case, locally, near zero the behavior of $f(\psi)$ remains the same both for $æ = 0$ and for $æ \neq 0$.

Let us evaluate the effect of the magnetic field inside a plasma on the flow pattern for a particular case where the field inside a plasma and the axial velocity component for $z = 0$ are not dependent on the radius. We assume that $w_0 = \text{const}$ and $a = H_0 R_{10} \nu$ (H_0 is the dimensionless axial component of the magnetic field in a plasma for $z = 0$). From (1.8), we then obtain

$$R = R_1(\psi)\nu, \qquad R_1 = w_0 R_{10}(w_0^2 + \psi)^{-1/2};$$
$$f(\psi) = R_1 + \Phi[\Phi^2(1 - R_{10})^{-2} - \psi(1 - H_0^2/w_0^2) - æ^2/R_1 + æ^2/R_{10}]^{-1/2}. \tag{2.2}$$

Clearly, two cases are possible.

(1) $1 - H_0^2/w_0^2 > 0$. Introducing $\Phi_* = \Phi(1 - H_0^2/w_0^2)^{-1/2}$ and $æ_* = æ(1 - H_0^2/w_0^2)^{-1/2}$, we reduce the problem to the preceding one, where Φ_* and $æ_*$ are used instead of Φ and $æ$, respectively. Thus, a sufficiently low internal field has no effect on the qualitative behavior of the flow.

(2) $1 - H_0^2/w_0^2 < 0$. Under this condition, to clarify up the behavior qualitatively, we confine ourselves to the case $æ = 0$. Here $f'(\psi) < 0$. The function η_0 increases with increasing $f(\psi)$, and, hence, ψ and w_0 decrease. If η_0 decreases, then vice versa. Thus, in this case, the flow is always subcritical. It follows that a sufficiently high field inside a plasma affects strongly the flow pattern.

3. Discussion of Results. Let us dwell upon the most considerable qualitative difference between the flows in nozzles with a vacuum gap and without it. We consider the conditions under which a subcritical flow regime is realized.

The subcriticality condition is $f'(0) < 0$. Expressing f according to (2.2) and assuming that the field inside a plasma is sufficiently low, i.e., the case $1 - H_0^2/w_0^2 > 0$ is realized, we find that the inequality $f'(0) < 0$ is satisfied if

$$\frac{(1 - R_{10})^3}{\Phi_*^2} + \frac{æ_*^2(1 - R_{10})^3}{2R_{10}\Phi_*^2} \int_0^1 w_0^{-2} d\nu - R_{10} \int_0^1 w_0^{-2} d\nu < 0.$$

In going to dimensional variables, under the assumption that $w_0 = \text{const}$, we obtain

$$\frac{H_{*30}^2}{4\pi\rho w_0^2} > \left[\left(1 - \frac{H_{30}^2}{4\pi\rho w_0^2}\right) + \frac{H_{*20}^2}{8\pi\rho w_0^2}\right]\left(\frac{r_{00}^2}{r_{10}^2} - 1\right), \tag{3.1}$$

where H_{*30} and H_{*20} are the axial and azimuthal components of the magnetic field in the gap for $z = 0$ and $r = r_{10}$, H_{30} is the axial component of the magnetic field inside a plasma for $z = 0$, r_{10} and r_{00} are the radii of the free surfaces of the pinch and nozzle for $z = 0$, respectively, w_0 is the dimensional axial component of the plasma velocity for $z = 0$, and ρ is the plasma density. For a supercritical regime, the conditions are found similarly.

It follows from (3.1) that the flow regime is determined not only by the relationship between the magnitude of the magnetic field and the axial velocity, but also by the flow geometry (the ratio of r_{00} to r_{10}). There is a considerable difference in the flows with a vacuum gap near the walls and the flows immediately adjacent to them. Thus, with the same relationship between the magnetic field and the axial velocity component, varying the ratio of r_{00} to r_{10}, one can transform a subscritical flow into a supercritical one and vice versa.

Note that according to the proposition, the plasma motion in a subcritical or supercritical regime is qualitatively similar to the flows of an ideal gas in nozzles. For example, the gas in a convergent nozzle accelerates in a subcritical flow and decelerates in a divergent one, and vice versa in a supercritial regime. Thus, the gas flow depends on the ratio of its velocity to the velocity of sound. In the case considered, the flow character depends on the ratio between the flow velocity and the velocity of long waves propagating over the plasma surface.

Let us discuss the applicability and importance of the model considered. The choice of the model of an incompressible plasma was made for the following reasons:

(1) It is possible to construct a rigorous easy-to-interpret theory and to study correctly, on its basis, the effects that are inherent in incompressible flows and that will show up, in one form or another, in more complicated cases as well;

(2) Based on the results obtained, one can make a more complicated analysis (for example, to incorporate the compressibility). Thus, the results presented can serve as the foundation for further advance of the theory.

In addition, the very model of an incompressible plasma is of interest. It is applicable not only to slow flows but to fast ones if the plasma pressure varies slightly during its motion in a nozzle. Since the plasma pressure is determined by the magnetic pressure in a gap, the pressure varies little if the vacuum gap changes not too significantly during the entire period of plasma motion.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 96-01-01771).

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